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A note on generalized and hypergeneralized projectors

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Abstract

Groß and Trenkler [Generalized and hypergeneralized projectors, Linear Algebra Appl. 264 (1997) 463–474] have introduced two generalizations of orthogonal projectors called generalized projectors and hypergeneralized projectors. In this note we characterize these generalizations by their spectral decompositions.

Keywords: Projection; Generalized projection; Spectral decomposition

A matrix A with real or complex elements satisfying $A^2 = A$ is said to be idempotent. It is well known that an idempotent matrix can be represented in the form

$$A = UV^*, \text{ where } U^*V = I. \tag{1}$$

If x is vector, Ax is in the column space $\mathscr{R}(U)$ of U. Moreover, Ax is stripped of the component of x in the orthogonal complement $\mathscr{R}(V)_\perp$ of the column space of V. For this reason A is also called the projector onto $\mathscr{R}(U)$ along $\mathscr{R}(V)_\perp$.

When A is Hermitian ($A = A^*$), we may take $U = V$, in which case U is orthonormal. We say that A is an orthogonal projector. When it is necessary to distinguish

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an orthogonal projector from a projector that is not orthogonal, the latter is called an oblique projector.

In [4], Groß and Trenkler introduced two generalizations of projectors.[1] Specifically, they defined a matrix A to be a generalized projector if

$$A^2 = A^*. \tag{2}$$

They call A a hypergeneralized projector if

$$A^2 = A^\dagger, \tag{3}$$

where $A^\dagger$ is the Moore–Penrose generalized inverse, characterized by the following Penrose conditions (e.g., see [6, Section 3.1]):

1. $A^\dagger A A^\dagger = A^\dagger$
2. $A A^\dagger A = A$
3. $A A^\dagger = (A A^\dagger)^*$
4. $A^\dagger A = (A^\dagger A)^*$

Groß and Trenkler derive some properties of generalized and hypergeneralized projectors. In a subsequent paper Baksalary et al. [2] derive some further properties. Although Groß and Trenkler give decompositional characterizations of their generalizations, the characterizations do not give a clear picture of what the generalizations are. In this paper we remedy this defect by exhibiting the spectral decompositions of the generalizations.[2]

The spectral decomposition we will need can be described as follows. A diagonalizable matrix A whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_m$ can be written in the form

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m,$$

where the P_i are projectors satisfying $P_1 + \cdots + P_m = I$ and $P_i P_j = 0$ $(i \neq j)$. If $P_i = U_i V_i^*$, as in (1), then the columns of U_i span the right eigenspace (aka invariant subspace) corresponding to λ_i, and the columns of V_i span the corresponding left eigenspace. If the left and right eigenspaces are the same, then P_i is an orthogonal projector. All this follows from the material in [5, Section 4.1].

We begin by establishing the spectral decomposition of a generalized projector. From (2) we have

$$AA^* = A^3 = A^*A.$$

[1] Our nomenclature is different from theirs. By projector Groß and Trenkler mean what we call an orthogonal projector, and they have no expression other than idempotent matrix for what we here call a projector. Since the distinction between orthogonal and oblique projectors is critical to this note, it seems reasonable to let projector be the general term and qualify it with orthogonal or oblique as required—a convention which has been around for almost half a century: see [1].

[2] The characterization (4) of generalized projectors has been given independently by Du and Li [3].

Hence A is an normal matrix. Such a matrix is diagonalizable and the left and right eigenspaces corresponding to a eigenvalue are the same. Again by (2) the eigenvalues of A must satisfy

$$\lambda^2 = \bar{\lambda}.$$

The only numbers satisfying this equation are 0, 1, ω, and ω^2, where

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

is the primitive cube root of unity. Thus our generalized projector can be written in the form

$$A = \omega^0 P_0 + \omega^1 P_1 + \omega^2 P_2 + 0 \cdot Q, \tag{4}$$

where P_0, P_1, P_2, and Q are orthogonal projectors with Q projecting onto the null space of A.

Conversely, if A has the representation (4), then A is a generalized projector. For it is easily verified that

$$A^2 = \omega^0 P_0 + \omega^2 P_1 + \omega^1 P_2 + 0 \cdot Q = A^*.$$

Turning now to hypergeneralized projectors, we use the fact that $\mathscr{R}(A^\dagger) = \mathscr{R}(A^*)$. From (3), it follows that

$$\mathscr{R}(A) \supset \mathscr{R}(A^2) = \mathscr{R}(A^*).$$

Since $\mathscr{R}(A)$ and $\mathscr{R}(A^*)$ have the same dimension, we must have $\mathscr{R}(A) = \mathscr{R}(A^*)$. It follows that A and A^* also have the same null space, which is orthogonal to $\mathscr{R}(A)$.

Let $X = (X_A \quad X_\perp)$ be a unitary matrix with X_A spanning $\mathscr{R}(A)$ and hence with $X_\perp$ spanning the null space of A. Then

$$X^* A X = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

where B is nonsingular. Now from the Penrose condition $AA^\dagger A = A$ and (3), we have

$$A = AA^\dagger A = AA^2 A = A^4. \tag{5}$$

It follows that $B^4 = B$ or $B^3 = I$. Since the Jordan structure of a nonzero eigenvalue is not changed by powering its matrix and B^3 is diagonalizable, it follows that B and hence A are diagonalizable. Moreover, (5) implies that the eigenvalues of A can only be 0, 1, ω, and ω^2, where as above ω is the primitive cube root of unity. It follows that

$$A = \omega^0 P_0 + \omega^1 P_1 + \omega^2 P_2 + 0 \cdot Q, \tag{6}$$

where P_i $(i = 1, 2, 3)$ are projectors and Q is an orthogonal projector. Note that although the P_i may oblique, they are not arbitrary, since

$$A^3 = P_0 + P_1 + P_2 = I - Q \tag{7}$$

is an orthogonal projector.

It is straightforward to verify that if A has the form (6), then A^2 satisfies the Penrose conditions. Thus the representation (6) is a characterization of hypergeneralized projectors.

It is curious that the characterizations of generalized projectors and hypergeneralized projectors have the same form, the difference being that the latter admits oblique projectors. Note that although these matrices generalize orthogonal projectors in the sense that that satisfy (2) or (3), both of which are satisfied by orthogonal projectors, they do not define new classes of projectors. For if P_1 and P_2 are zero, which is necessary for A to be a projector, then by (7) P_0, and hence A, must be orthogonal (see also [4, Fig. 1]).

Finally, we note that the above arguments generalize to the cases

$$A^k = A^* \quad \text{and} \quad A^k = A^\dagger.$$

The characterizing spectral decompositions become

$$A = \omega^0 P_0 + \omega^1 P_1 + \cdots + \omega^k P_k + 0 \cdot Q,$$

where ω is now the primitive $(k+1)$th root of unity.

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